BASICS FROM REPRESENTATION THEORY

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Our goal here is to introduce representation theory for finite groups, that is the study of actions of finite groups on vector spaces.

1. Definitions and examples

Recall first that an *action* of a group G on a set X is a group homomorphism $G \longrightarrow S(X)$, where the latter is the group of bijective maps (called *permutations*) of X.

Definition 1.1. Let G be a finite group, and V be a finite-dimensional vector space. A *representation* of G on V is a group homomorphism

$$\varphi\colon G \longrightarrow \mathrm{GL}(V).$$

The dimension of V is then called the degree of φ, and is denoted $\deg(\varphi)$.

Given a representation of G on V, we usually write φ_g for $\varphi(g)$, and $\varphi_g(v)$, or $\varphi_g v$, for $\varphi(g)(v)$.

Example 1.2. Any group G has a degree one trivial representation, defined by $\varphi\colon G \longrightarrow \mathbb{C}^*, g \longmapsto 1$.

Example 1.3. The map $\mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{C}^*$, $[k] \longmapsto (-1)^k$, is a degree one representation of $\mathbb{Z}/2\mathbb{Z}$.

Example 1.4. More generally, the map $\mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{C}^*$, $[k] \longmapsto e^{\frac{2\pi i k}{n}}$, is a degree one representation of $\mathbb{Z}/n\mathbb{Z}$.

Example 1.5. Define $\varphi\colon S_n \longrightarrow \mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}(\mathbb{C}^n)$ on the standard basis of $\mathbb{C}^n$ via $\varphi_\sigma(e_i) = e_{\sigma(i)}$, $\sigma \in S_n$, $1 \leq i \leq n$. Given $\sigma \in S_n$, the matrix of φ_σ in the standard basis of $\mathbb{C}^n$ is obtained by permuting the rows of I_n according to σ. For instance, if $n = 3$, then

$$\varphi_{(13)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \varphi_{(132)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ be a representation of degree n. Given a basis B of V, we can associate a vector space isomorphism $T\colon V \longrightarrow \mathbb{C}^n$, and define another representation ψ of G on $\mathbb{C}^n$ by $\psi_g = T\varphi_g T^{-1}$, $g \in G$. We want to think of this action of G on $\mathbb{C}^n$ as being the same as its initial action on V. This motivates the next definition.

Definition 1.6. Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ and $\psi\colon G \longrightarrow \mathrm{GL}(W)$ be two representations of G. We say φ and ψ are *equivalent*, and we denote $\varphi \sim \psi$, if there exists a vector spaces isomorphism $T\colon V \longrightarrow W$ so that $T\varphi_g = \psi_g T$ for all $g \in G$.

It is straightforward to check $\sim$ is reflexive, symmetric and transitive.

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Example 1.7. Define $\varphi\colon \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ and $\psi\colon \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ by

$$\varphi_{[k]} = \begin{pmatrix} \cos(2\pi\frac{k}{n}) & -\sin(2\pi\frac{k}{n}) \\ \sin(2\pi\frac{k}{n}) & \cos(2\pi\frac{k}{n}) \end{pmatrix}, \quad \psi_{[k]} = \begin{pmatrix} e^{\frac{2\pi i k}{n}} & 0 \\ 0 & e^{-\frac{2\pi i k}{n}} \end{pmatrix}.$$

Then $\varphi \sim \psi$, and a direct computation shows that $\varphi_{[k]} = A\psi_{[k]}A^{-1}$ for any $0 \le k \le n-1$, where $A = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$.

Note that in Example 1.5 we have

$$\varphi_\sigma(e_1 + \cdots + e_n) = e_{\sigma(1)} + \cdots + e_{\sigma(n)} = e_1 + \cdots + e_n$$

for any $\sigma \in S_n$. This means that the subspace $\mathbb{C}(e_1 + \cdots + e_n)$ is invariant (in fact even fixed) under the action of φ_σ, and restricting our attention to this subspace provides a new action (here trivial) of the group on a vector space.

Definition 1.8. Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ be a representation. A subspace $W \subset V$ is called $G-$*invariant* if $\varphi_g w \in W$ for any $g \in G$ and any $w \in W$.

For ψ as in Example 1.7, $\mathbb{C}e_1$ and $\mathbb{C}e_2$ are both $\mathbb{Z}/n\mathbb{Z}-$invariant and $\mathbb{C} = \mathbb{C}e_1 \oplus \mathbb{C}e_2$. This is not a special feature of this representation, and we will prove later that such a decomposition always exists.

Definition 1.9. Let $\varphi^{(1)}\colon G \longrightarrow \mathrm{GL}(V_1)$, $\varphi^{(2)}\colon G \longrightarrow \mathrm{GL}(V_2)$ be two representations of G. Their *direct sum* is the representation $\varphi^{(1)} \oplus \varphi^{(2)}\colon G \longrightarrow \mathrm{GL}(V_1 \oplus V_2)$ defined as

$$(\varphi^{(1)} \oplus \varphi^{(2)})_g(v_1, v_2) := (\varphi^{(1)}_g v_1, \varphi^{(2)}_g v_2)$$

for any $g \in G$ and $(v_1, v_2) \in V_1 \oplus V_2$.

In terms of matrices, if $\varphi^{(1)}\colon G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ and $\varphi^{(2)}\colon G \longrightarrow \mathrm{GL}_m(\mathbb{C})$, then $\varphi^{(1)} \oplus \varphi^{(2)}\colon G \longrightarrow \mathrm{GL}_{n+m}(\mathbb{C})$ and

$$(\varphi^{(1)} \oplus \varphi^{(2)})_g = \begin{pmatrix} \varphi^{(1)}_g & 0 \\ 0 & \varphi^{(2)}_g \end{pmatrix}, \ g \in G.$$

Example 1.10. The representation ψ of Example 1.7 is the direct sum $\varphi^{(1)} \oplus \varphi^{(2)}$, where $\varphi^{(1)}_{[k]} = e^{\frac{2\pi i k}{n}}$ and $\varphi^{(2)}_{[k]} = e^{-\frac{2\pi i k}{n}}$, $0 \le k \le n-1$.

Example 1.11. If $n > 1$ and φ is the representation of G given by $\varphi_g = I_n$ for any $g \in G$, φ is *not* the trivial representation of G but rather the direct sum of n copies of the trivial representation.

Example 1.12. Let $\varphi\colon S_3 \longrightarrow \mathrm{GL}_2(\mathbb{C})$ be defined on the generators (12) and (123) of S_3 as

$$\varphi_{(12)} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \varphi_{(123)} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

and let ψ be the trivial representation of S_3. Then

$$(\varphi \oplus \psi)_{(12)} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\varphi \oplus \psi)_{(123)} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In fact we will see later that this representation is equivalent to the one of Example 1.5.

If $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ is a representation and $W \subset V$ is G−invariant, one may restrict φ to W to get a well-defined representation $\varphi|_W\colon G \longrightarrow \mathrm{GL}(W)$. We then say that $\varphi|_W$ is a *subrepresentation* of φ. If $V_1, V_2 \subset V$ are G−invariant and $V = V_1 \oplus V_2$, then φ is equivalent to the direct sum $\varphi|_{V_1} \oplus \varphi|_{V_2}$.

In mathematics, it is often the case that one has a sort of factorization into primes or irreducibles. This also happens in representation theory.

Definition 1.13. A representation $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ is called *irreducible* if the only G−invariant subspaces of V are $\{0\}$ and V.

Example 1.14. Any one dimensional representation $\varphi\colon G \longrightarrow \mathbb{C}^*$ is irreducible, as the only subspaces of $\mathbb{C}$ are $\{0\}$ and $\mathbb{C}$ itself.

Example 1.15. As already mentioned, the representation ψ of Example 1.7 has two non-trivial invariant subspaces, and is therefore not irreducible. Likewise, using the matrix A of the same example, one can deduce that

$$\mathbb{C}\begin{pmatrix} i \\ 1 \end{pmatrix}, \quad \mathbb{C}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

are non-trivial invariant subspaces for φ, which is also not irreducible.

We will establish below that being irreducible is an invariant of equivalences of representations. Non-irreducibility of ψ therefore directly implies non-irreducibility of φ.

In fact, we also could deduce that $\mathbb{C}\begin{pmatrix} i \\ 1 \end{pmatrix}$ is invariant for φ by noticing the latter is actually an eigenspace for each $\varphi_{[k]}$. Let us work out this idea on another example before stating a general result.

Example 1.16. Consider the representation of $G = S_3$ given by

$$\varphi_{(12)} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \varphi_{(123)} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

We claim this representation is irreducible.

Proof. Indeed, suppose towards a contradiction that $W \subset \mathbb{C}^2$ is a non-trivial S_{3-}invariant subspace. Then $\dim(W) = 1$. Fix $v \in W, v \neq 0$, so that $W = \mathbb{C}v$. As $\varphi_{(12)}v, \varphi_{(123)}v \in W$, we have $\varphi_{(12)}v = \lambda v$, $\varphi_{(123)}v = \mu v$ for some $\lambda, \mu \in \mathbb{C}$. This implies v is a common eigenvector for $\varphi_{(12)}$ and $\varphi_{(123)}$. On the other hand, a direct computation shows that the eigenvalues of $\varphi_{(12)}$ are 1 and −1, with corresponding eigenspaces

$$E_1 = \mathbb{C}\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad E_{-1} = \mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and neither $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ nor $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector for $\varphi_{(123)}$. This is a contradiction, and thus φ is irreducible. □

Our first proposition is then a criterion for irreducibility.

Proposition 1.17. *Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ be a representation of G of degree 2 or 3. Then φ is irreducible if and only if there is no common eigenvector to all φ_g, $g \in G$.*

2. Maschke's theorem and complete reducibility

Our goal is to show that any representation is equivalent to a direct sum of irreducible representations. To that aim, we adopt some terminologies.

Definition 2.1. Let G be a group. A representation $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ is said to be *completely reducible* if $V = V_1 \oplus \cdots \oplus V_n$, where $V_1, \ldots, V_n$ are $G-$invariant subspaces of V and $\varphi|_{V_i}$ is irreducible for all $1 \leq i \leq n$.

Equivalently, φ is completely reducible if $\varphi \sim \varphi^{(1)} \oplus \cdots \oplus \varphi^{(n)}$, where $\varphi^{(i)}$ is irreducible for all $1 \leq i \leq n$.

Definition 2.2. A representation $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ is called *decomposable* if there exists non-trivial $G-$invariant subspaces V_1, V_2 so that $V = V_1 \oplus V_2$. Otherwise, φ is called *indecomposable*.

First, we show that these notions are invariant under equivalences of representations.

Proposition 2.3. *Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ be a representation equivalent to a decomposable representation $\psi\colon G \longrightarrow \mathrm{GL}(W)$. Then φ is decomposable.*

Proof. As $\varphi \sim \psi$, let $T\colon V \longrightarrow W$ be an isomorphism of vector spaces so that $\psi_g T = T\varphi_g$ for all $g \in G$. By assumption, we may find $W_1, W_2 \subset W$ two non-trivial $G-$invariant subspaces so that $W = W_1 \oplus W_2$. Let then $V_1 := T^{-1}(W_1)$, $V_2 := T^{-1}(W_2)$.

First, observe that $V_1 \cap V_2 = T^{-1}(W_1 \cap W_2) = T^{-1}(\{0\}) = \{0\}$ as T is an isomorphism. Also if $v \in V$, then $Tv \in W$, so there exists a unique pair $w_1 \in W_1$, $w_2 \in W_2$ so that $Tw = w_1 + w_2$. Now $v = T^{-1}w_1 + T^{-1}w_2 = v_1 + v_2$ with $v_1 \in V_1$, $v_2 \in V_2$, and the pair we just found is unique since T is an isomorphism. This proves that $V = V_1 \oplus V_2$.

It remains to prove that V_1, V_2 are $G-$invariant. For instance, let $v \in V_1$ and $g \in G$. Then

$$\varphi_g v = T^{-1}\psi_g T v$$

and $Tv \in W_1$ which is $G-$invariant, so $\psi_g Tv \in W_1$, whence $\varphi_g v = T^{-1}\psi_g Tv \in T^{-1}(W_1) = V_1$. Thus V_1 is $G-$invariant, and the same reasoning shows V_2 is $G-$invariant as well. □

A similar proof shows analoguous statements for irreducibility and complete reducibility.

Proposition 2.4. *Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ be a representation equivalent to a completely reducible (resp. irreducible) representation $\psi\colon G \longrightarrow \mathrm{GL}(W)$. Then φ is completely reducible (resp. irreducible).*

The strategy for proving a decomposition into a direct sum of irreducible representations is to prove that each representation is either irreducible or decomposable, and then proceed by induction on the degree. This last fact could seem obvious, but may fail for representations of infinite groups. Here is an example.

Example 2.5. Let $\varphi\colon \mathbb{Z} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ be given by

$$\varphi(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

It is easy to check φ is a homomorphism, and that e_1 is an eigenvector for all $\varphi(n)$, $n \in \mathbb{Z}$. Thus φ is not irreducible. On the other hand, if φ is decomposable, it is equivalent to a direct sum of two one-dimensional representations. Such a representation is diagonal, and thus $\varphi(1)$ is diagonalisable, a contradiction. Hence φ is indecomposable.

To prove every representation of a finite group is either irreducible or decomposable, we will proceed in two steps, first showing this holds for *unitary* representations, and then showing every representation is equivalent to a unitary one.

Definition 2.6. Let V be an inner product space. A representation $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ is *unitary* if

$$\langle \varphi_g u, \varphi_g v\rangle = \langle u, v\rangle$$

for any $g \in G$ and $u, v \in V$.

In other words, a representation φ is unitary if its image lies into $U(V)$, the unitary group of V.

Example 2.7. Let $\varphi\colon \mathbb{R} \longrightarrow \mathbb{T}, t \longmapsto e^{2\pi it}$. Then $\varphi(t+s) = e^{2\pi i(t+s)} = e^{2\pi it}e^{2\pi is} = \varphi(t)\varphi(s)$ for all $s, t \in \mathbb{R}$, so φ is a homomorphism, and as $\mathbb{T} = U(\mathbb{C})$, it is a one-dimensional unitary representation of $\mathbb{R}$.

We can now proceed to show the dichotomy announced above for unitary representations.

Proposition 2.8. *Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ be a unitary representation of a finite group G. Then φ is either irreducible or decomposable.*

Proof. If φ is irreducible, there is nothing to prove. Suppose then it is not, and let $W \subset V$ be a non-trivial $G-$invariant subspace. Then its orthogonal $W^\perp$ is also a proper subspace of V and $V = W \oplus W^\perp$. It remains to prove it is also $G-$invariant. Let $g \in G$ and $v \in W^\perp$. Then for $w \in W$ one has

$$\langle \varphi_g v, w\rangle = \langle \varphi_{g^{-1}}\varphi_g v, \varphi_{g^{-1}} w\rangle = \langle v, \varphi_{g^{-1}} w\rangle$$

since φ_g is unitary. As W is $G-$invariant, $\varphi_{g^{-1}} w \in W$ and since $v \in W^\perp$, we deduce $\langle v, \varphi_{g^{-1}} w\rangle = 0$. Hence $\varphi_g v \in W^\perp$, and the latter is therefore $G-$invariant. This proves that φ is decomposable. □

The second step is then to prove any representation is equivalent to a unitary representation.

Proposition 2.9. *Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ be a representation of a finite group G. Then φ is equivalent to a unitary representation.*

Proof. Let $n := \dim(V)$ and fix an isomorphism of vector spaces $T\colon V \longrightarrow \mathbb{C}^n$. Setting $\rho_g = T\varphi_g T^{-1}$ for any $g \in G$ yields a representation $\rho\colon G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ equivalent to φ. As $\sim$ is an equivalence relation, it is then enough to prove that ρ is equivalent to a unitary representation. To this aim, we show that there is an inner product on $\mathbb{C}^n$ for which ρ is unitary. Let $\langle\cdot,\cdot\rangle$ be the standard inner product on $\mathbb{C}^n$, and define

$$(u, v) := \sum_{g\in G} \langle \rho_g u, \rho_g v\rangle,\ u, v \in \mathbb{C}^n.$$

It is straightforward to check $(\cdot,\cdot)$ is an inner product. For instance if $u \in \mathbb{C}^n$ one has

$$(u, u) = \sum_{g\in G} \langle \rho_g u, \rho_g u\rangle$$

and $\langle \rho_g u, \rho_g u\rangle \geq 0$ for all $g \in G$, whence $(u,u) \geq 0$. Also if $(u,u) = 0$ then $\langle \rho_g u, \rho_g u\rangle = 0$ for all $g \in G$. In particular, for $g = e$, we get $\langle u, u\rangle = 0$, whence $u = 0$. Additionally, ρ is unitary for this new inner product, as

$$(\rho_h u, \rho_h v) = \sum_{g\in G} \langle \rho_g \rho_h u, \rho_g \rho_h v\rangle = \sum_{g\in G} \langle \rho_{gh} u, \rho_{gh} v\rangle = \sum_{t\in G} \langle \rho_t u, \rho_t v\rangle = (u,v)$$

since $g \longmapsto gh$ is a permutation of G. To conclude, choose $\{f_1, \ldots, f_n\}$ an orthonormal basis of $(\mathbb{C}^n, (\cdot,\cdot))$, define an isomorphism $S\colon (\mathbb{C}^n, \langle\cdot,\cdot\rangle) \longrightarrow (\mathbb{C}^n, (\cdot,\cdot))$ by $Se_i = f_i$, for $1 \leq i \leq n$, and extend it by linearity. Set $T := \mathrm{Id}_{\mathbb{C}^n} \circ S$, where $\mathrm{Id}_{\mathbb{C}^n}\colon (\mathbb{C}^n, (\cdot,\cdot)) \longrightarrow (\mathbb{C}^n, \langle\cdot,\cdot\rangle)$. The invariance of $(\cdot,\cdot)$ with respect to ρ now implies that $T\rho_g T^{-1}$ is unitary for $\langle\cdot,\cdot\rangle$, for any $g \in G$. This concludes the proof. □

As explained earlier, Propositions 2.3, 2.4, 2.8, 2.9 together provide the following corollary.

Corollary 2.10. *Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ be a representation of a finite group G. Then φ is either irreducible or decomposable.*

Remark 2.11. Clearly, irreducible representations are indecomposable. Example 2.5 shows the converse may fail.

Now we know every representation is either irreducible or decomposable, we proceed to establish Maschke's theorem, the central result of this part.

Theorem 2.12. *Any representation of a finite group is completely reducible.*

Proof. Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ be a representation of a finite group G. We prove the statement by induction on n, the degree of the representation. If $n = 1$, φ is irreducible by Example 1.14. Suppose then $n > 1$, and that the statement holds for any representation of a finite group of degree at most $n-1$. Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ be a representation of degree n. If φ is irreducible, we are done. If not, it is decomposable by Corollary 2.10, and we find $V_1, V_2 \subset V$ non-trivial $G-$invariant subspaces of V so that $V = V_1 \oplus V_2$. As $\dim(V_1), \dim(V_2) < n$, the induction hypothesis implies that $\varphi|_{V_1}$ and $\varphi|_{V_2}$ are completely reducible, that is there exist $U_1, \ldots, U_s, W_1, \ldots, W_r$ which are $G-$invariant so that

$$V_1 = U_1 \oplus \cdots \oplus U_s,\ V_2 = W_1 \oplus \cdots \oplus W_r$$

and $\varphi|_{U_i}, \varphi|_{W_j}$ are irreducible, for any $1 \leq i \leq s, 1 \leq j \leq r$. Then

$$V = V_1 \oplus V_2 = U_1 \oplus \cdots \oplus U_s \oplus W_1 \oplus \cdots \oplus W_r$$

and φ is completely reducible. This achieves the inductive step and the proof. □

Naturally, the question that arises with this result is about the (non-)uniqueness of the decomposition of a representation into irreducible ones. This will be solved in a next section.

3. Schur's lemma and orthogonality relations

We now turn to study the "morphisms" of the theory.

Definition 3.1. Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$, $\rho\colon G \longrightarrow \mathrm{GL}(W)$ be representations of a group G. A *morphism from φ to ρ* is a linear map $T\colon V \longrightarrow W$ so that $T\varphi_g = \rho_g T$ for any $g \in G$.

This notion is a weakening of equivalences of representations, since we do not require T to be invertible.

For two representations φ, ρ, we denote $\mathrm{Hom}_G(\varphi, \rho)$ the set of morphisms from φ to ρ. Obviously, $\mathrm{Hom}_G(\varphi, \rho) \subset \mathrm{Hom}(V, W)$. An element $T \in \mathrm{Hom}_G(\varphi, \rho)$ is often called an *intertwiner*, or *intertwining operator*, of φ and ρ.

An immediate consequence of the previous definition is the linear structure of $\mathrm{Hom}_G(\varphi, \rho)$.

Proposition 3.2. *Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$, $\rho\colon G \longrightarrow \mathrm{GL}(W)$ be representations of a group G. Then $Hom_G(\varphi, \rho)$ is a subspace of $Hom(V, W)$.*

Proof. Clearly, $0 \in \mathrm{Hom}_G(\varphi, \rho)$, and if $T_1, T_2 \in \mathrm{Hom}_G(\varphi, \rho)$ and $c \in \mathbb{C}$, then

$$(T_1 + cT_2)\varphi_g = T_1\varphi_g + cT_2\varphi_g = \rho_g T_1 + c\rho_g T_2 = \rho_g(T_1 + cT_2)$$

for any $g \in G$, so that $T_1 + cT_2 \in \mathrm{Hom}_G(\varphi, \rho)$ as well. □

The next proposition is also a direct consequence of the definition.

Proposition 3.3. *If $T\colon V \longrightarrow W$ is in $Hom_G(\varphi, \rho)$, then $Ker(T)$ and $Im(T)$ are $G-$invariant.*

Proof. First, suppose $v \in \mathrm{Ker}(T)$. Then, since $T\varphi_g = \rho_g T$ for any $g \in G$, it follows that

$$T\varphi_g v = \rho_g T v = \rho_g(0) = 0$$

for any $g \in G$, thus $\varphi_g v \in \mathrm{Ker}(T)$, which is $G-$invariant.

Now, if $w \in \mathrm{Im}(T)$, say $w = Tv$, then

$$\rho_g w = \rho_g T v = T\varphi_g v \in \mathrm{Im}(T)$$

for any $g \in G$, whence $\mathrm{Im}(T)$ is $G-$invariant. □

The next result, usually referred to as the *Schur's lemma*, is fundamental to representation theory. Roughly speaking, it says that morphisms between irreducible representations are very limited.

Lemma 3.4. *Let φ, ρ be irreducible representations of a group G, and let $T \in Hom_G(\varphi, \rho)$. Then either $T = 0$ or T is an isomorphism. Consequently:*

(i) If $\varphi \nsim \rho$, then $Hom_G(\varphi, \rho) = \{0\}$.
(ii) If $\varphi = \rho$, there exists $\lambda \in \mathbb{C}$ so that $T = \lambda Id_V$.

Proof. Let then $\varphi\colon G \longrightarrow \mathrm{GL}(V)$, $\rho\colon G \longrightarrow \mathrm{GL}(W)$ be irreducible representations of G, with $T \in \mathrm{Hom}_G(\varphi, \rho)$. If $T = 0$, we are done.

Assume now $T \neq 0$. From Proposition 3.3, $\mathrm{Ker}(T)$ is a $G-$invariant subspace of V, whence $\mathrm{Ker}(T) = \{0\}$ or $\mathrm{Ker}(T) = V$ by irreducibility of φ. As $T \neq 0$ this case is excluded, so $\mathrm{Ker}(T) = \{0\}$ and T is injective. As also $\mathrm{Im}(T)$ is $G-$invariant and $T \neq 0$, it follows that $\mathrm{Im}(T) = W$, and T is surjective. In conclusion, T is an isomorphism.

For *(i)*, suppose $\mathrm{Hom}_G(\varphi, \rho) \neq \{0\}$, and pick $T \neq 0$ a morphism between φ and ρ. As we just proved, T is an isomorphism, whence $\varphi \sim \rho$.

For *(ii)*, suppose that $T\colon V \longrightarrow V \in \mathrm{Hom}_G(\varphi, \varphi)$, and let $\lambda \in \mathbb{C}$ be an eigenvalue of T. Then $T - \lambda\mathrm{Id}_V$ is not invertible, and from Proposition 3.2 one has also $T - \lambda\mathrm{Id}_V \in \mathrm{Hom}_G(\varphi, \varphi)$. From the first part of the proof, we deduce $T - \lambda\mathrm{Id}_V = 0$, *i.e.* $T = \lambda\mathrm{Id}_V$. □

We are now able to describe irreducible representations of abelian groups.

Corollary 3.5. *Let G be an abelian group. Then any irreducible representation of G has degree one.*

Proof. Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ be an irreducible representation of G. Fix $h \in G$. Then

$$\varphi_h\varphi_g = \varphi_{hg} = \varphi_{gh} = \varphi_g\varphi_h$$

for any $g \in G$, and Schur's lemma then implies there is $\lambda_h \in \mathbb{C}$ so that $\varphi_h = \lambda_h \mathrm{Id}_V$. If $v \neq 0 \in V$ and $\lambda \in \mathbb{C}$, then

$$\varphi_h\lambda v = \lambda_h\lambda v \in \mathbb{C}v$$

so is a $G-$invariant subspace of V. From the irreducibility of φ, it follows that $V = \mathbb{C}v$, and therefore $\dim(V) = 1$, as announced. □

This result has nice applications in linear algebra.

Corollary 3.6. *Let $\varphi\colon G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ be a representation of a finite abelian group G. Then there exists an invertible matrix $T \in \mathrm{GL}_n(\mathbb{C})$ so that $T^{-1}\varphi_g T$ is diagonal for all $g \in G$.*

Proof. By Theorem 2.12, φ is completely reducible, so $\varphi \sim \varphi^{(1)} \oplus \cdots \oplus \varphi^{(m)}$ where $\varphi^{(i)}$ is an irreducible representation of G, $1 \leq i \leq m$. Since G is abelian, $\varphi^{(i)}$ is one-dimensional, so $n = m$ and $\varphi_g^{(i)} \in \mathbb{C}^*$ for any $g \in G$ and $1 \leq i \leq n$. Denoting $T\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ the isomorphism realising the equivalence between φ and $\varphi^{(1)} \oplus \cdots \oplus \varphi^{(m)}$, it follows that $T^{-1}\varphi_g T$ is the diagonal $n \times n$ matrix whose $i-$th coefficient on the diagonal is $\varphi_g^{(i)}$, for any $g \in G$ and $1 \leq i \leq n$. □

We deduce from this result the diagonalisability of any matrix with finite order.

Corollary 3.7. *Let $A \in \mathrm{GL}_m(\mathbb{C})$ having finite order. Then A is diagonalisable, and its eigenvalues are $n-$th roots of unity, where $A^n = I_m$.*

Proof. Suppose then that $A^n = I_m$. Define a representation of $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{C}^m$ by

$$\varphi\colon \mathbb{Z}/n\mathbb{Z},\ [k] \longmapsto A^k.$$

It is easy to check that φ is a representation. As $\mathbb{Z}/n\mathbb{Z}$ is abelian, Corollary 3.6 implies there is $T \in \mathrm{GL}_m(\mathbb{C})$ so that $T^{-1}\varphi_{[1]}T = T^{-1}AT$ is diagonal, whence A is diagonalisable.

Now $T^{-1}AT = D$ is the matrix of the eigenvalues of $\lambda_1, \ldots, \lambda_m$ of A, and as

$$D^n = (T^{-1}AT)^n = T^{-1}A^nT = T^{-1}I_mT = I_m$$

it follows that $\lambda_i^n = 1$ for any $1 \leq i \leq m$. Thus $\lambda_1, \ldots, \lambda_m$ are $n-$th roots of unity, as claimed. □

From now on, G always denote a finite group. Let $\varphi\colon G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ be a representation of G. Then $\varphi(g) = (\varphi_{ij}(g))_{1\leq i,j\leq n}$ where $\varphi_{ij}(g) \in \mathbb{C}$, and thus $\varphi_{ij} \in \mathbb{C}^G = \{f\colon G \longrightarrow \mathbb{C}\}$.

Definition 3.8. Let G be a group. We denote $L(G) := \mathbb{C}^G$ the $\mathbb{C}-$vector space of complex valued functions defined on G, endowed with the inner product

$$\langle f_1, f_2\rangle := \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)},\ f_1, f_2 \in \mathbb{C}^G.$$

We are now going to show that when φ is irreducible and unitary, the set $\{\varphi_{ij} : 1 \leq i, j \leq n\}$ forms an orthogonal set in $L(G)$.

Theorem 3.9. *Let $\varphi\colon G \longrightarrow U_n(\mathbb{C})$, $\rho\colon G \longrightarrow U_m(\mathbb{C})$ be inequivalent irreducible unitary representations of G. Then*

(i) $\langle \varphi_{ij}, \rho_{kl} \rangle = 0$ *for all* $1 \le i, j \le n$, $1 \le k, l \le m$.
(ii) $\langle \varphi_{ik}, \varphi_{jl} \rangle = \frac{1}{n}\delta_{ij}\delta_{kl}$.

The proof of these orthogonality relations requires some preparations. The first one provides a generic way of getting intertwiners of representations.

Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$, $\rho\colon G \longrightarrow \mathrm{GL}(W)$ be representations of G, and let $T \in \mathrm{Hom}(V, W)$. Define a new linear transformation $\overline{T} \in \mathrm{Hom}(V, W)$ by

$$\overline{T} := \frac{1}{|G|}\sum_{g\in G} \rho_{g^{-1}} T \varphi_g.$$

This procedure has the following properties.

Lemma 3.10. *Let* $\varphi\colon G \longrightarrow \mathrm{GL}(V)$, $\rho\colon G \longrightarrow \mathrm{GL}(W)$ *be representations of* G, *and let* $T \in \mathit{Hom}(V, W)$. *Then*

(i) $\overline{T} \in \mathit{Hom}_G(\varphi, \rho)$.
(ii) If $T \in \mathit{Hom}_G(\varphi, \rho)$, *then* $\overline{T} = T$.
(iii) The map $\mathit{Hom}(V, W) \longrightarrow \mathit{Hom}_G(\varphi, \rho)$, $T \longmapsto \overline{T}$ *is linear and surjective.*

Proof. (i) Obvisously, $\overline{T} \in \mathrm{Hom}(V, W)$ and if $g \in G$ we compute that

$$\begin{aligned}
\overline{T}\varphi_h &= \frac{1}{|G|}\sum_{g\in G} \rho_{g^{-1}} T \varphi_g \varphi_h \\
&= \frac{1}{|G|}\sum_{g\in G} \rho_{g^{-1}} T \varphi_{gh} = \frac{1}{|G|}\sum_{t\in G} \rho_{ht^{-1}} T \varphi_t \\
&= \rho_h \frac{1}{|G|}\sum_{t\in G} \rho_{t^{-1}} T \varphi_t \\
&= \rho_h \overline{T}
\end{aligned}$$

whence $\overline{T} \in \mathrm{Hom}_G(\varphi, \rho)$.

(ii) If T already intertwines φ and ρ, then $T\varphi_g = \rho_g T$ for any $g \in G$, and it follows that

$$\overline{T} = \frac{1}{|G|}\sum_{g\in G} \rho_{g^{-1}} T \varphi_g = \frac{1}{|G|}\sum_{g\in G} \rho_{g^{-1}} \rho_g T = T$$

as claimed.

(iii) The surjectivity follows from *(ii)*, and if $T_1, T_2 \in \mathrm{Hom}(V, W)$ and $c \in \mathbb{C}$, then

$$\begin{aligned}
\overline{T_1 + cT_2} &= \frac{1}{|G|}\sum_{g\in G} \rho_{g^{-1}} (T_1 + cT_2) \varphi_g \\
&= \frac{1}{|G|}\sum_{g\in G} \rho_{g^{-1}} T_1 \varphi_g + c\frac{1}{|G|}\sum_{g\in G} \rho_{g^{-1}} T_2 \varphi_g \\
&= \overline{T_1} + c\overline{T_2}
\end{aligned}$$

establishing linearity of $T \longmapsto \overline{T}$. $\square$

Then, from Schur's lemma, we can in fact deduce an explicit expression for this intertwiner.

Proposition 3.11. *Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$, $\rho\colon G \longrightarrow \mathrm{GL}(W)$ be irreducible representations of G, and let $T \in Hom(V, W)$. Then*

(i) If $\varphi \nsim \rho$, then $\overline{T} = 0$.
(ii) If $\varphi = \rho$, then $\overline{T} = \frac{Tr(T)}{deg(\varphi)} Id_V$.

Proof. Point *(i)* follows from *(i)* of Lemma 3.4. For *(ii)*, we use *(ii)* of Lemma 3.4 to deduce there is $\lambda \in \mathbb{C}$ so that $\overline{T} = \lambda \mathrm{Id}_V$. To determine λ, we compute $\mathrm{Tr}(\overline{T})$ in two different ways. On the one hand we have

$$\mathrm{Tr}(\overline{T}) = \mathrm{Tr}(\lambda \mathrm{Id}_V) = \lambda \mathrm{Tr}(\mathrm{Id}_V) = \lambda \dim(V) = \lambda \deg(\varphi)$$

while on the other hand the initial definition of $\overline{T}$ provides

$$\mathrm{Tr}(\overline{T}) = \mathrm{Tr}\left(\frac{1}{|G|} \sum_{g \in G} \varphi_{g^{-1}} T \varphi_g\right) = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(\varphi_{g^{-1}} T \varphi_g) = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(T) = \mathrm{Tr}(T).$$

Thus $\lambda = \frac{\mathrm{Tr}(\overline{T})}{\deg(\varphi)} = \frac{\mathrm{Tr}(T)}{\deg(\varphi)}$, which concludes the proof. □

Let $\varphi\colon G \longrightarrow_n (\mathbb{C})$ and $\rho\colon G \longrightarrow \mathrm{GL}_m(\mathbb{C})$. The map $\mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}_G(\varphi, \rho)$ from Lemma 3.10 is then a map from $M_{mn}(\mathbb{C})$ to a subspace of $M_{mn}(\mathbb{C})$, and it is natural to try to compute its matrix with respect to the standard basis of $M_{mn}(\mathbb{C})$. It turns out this matrix has a special form is φ and ρ are unitary.

To make this claim more precise, we need the following lemma about matrix multiplication. Recall that the standard basis of $M_{mn}(\mathbb{C})$ is $E_{11}, E_{12}, \ldots, E_{mn}$ where E_{ij} has its $(i, j)-$th coefficient equal to 1, and the others equal to 0.

Lemma 3.12. *Let $A \in M_{rm}(\mathbb{C})$, $B \in M_{ns}(\mathbb{C})$ and $E_{ki} \in M_{mn}(\mathbb{C})$. Then $(AE_{ki}B)_{lj} = a_{lk} b_{ij}$.*

Proof. Directly $(AE_{ki}B)_{lj} = \sum_{r,s} a_{lr} (E_{ki})_{rs} b_{sj}$ and the only non-zero term in this sum is when $r = k$ and $s = i$, giving the announced formula. □

We can now proceed to prove the following.

Proposition 3.13. *Let $\varphi\colon G \longrightarrow U_n(\mathbb{C})$ and $\rho\colon G \longrightarrow U_m(\mathbb{C})$ be unitary representations of G. Let $A = E_{ki} \in M_{mn}(\mathbb{C})$. Then $(\overline{A})_{lj} = \langle \varphi_{ij}, \rho_{kl} \rangle$.*

Proof. From the definition of $\overline{A}$ and Lemma 3.12, we have

$$(\overline{A})_{lj} = \frac{1}{|G|} \sum_{g \in G} (\rho_{g^{-1}} E_{ki} \varphi_g)_{lj} = \frac{1}{|G|} \sum_{g \in G} \rho_{lk}(g^{-1}) \varphi_{ij}(g).$$

As ρ is unitary, $\rho(g^{-1}) = \rho(g)^{-1} = \rho(g)^*$, so $\rho_{lk}(g^{-1}) = \overline{\rho_{lk}(g)}$, and it follows that

$$(\overline{A})_{lj} = \frac{1}{|G|} \sum_{g \in G} \overline{\rho_{kl}(g)} \varphi_{ij}(g) = \langle \varphi_{ij}, \rho_{kl} \rangle$$

as was to be shown. □

We have all we need to establish Schur's orthogonality relations, namely Theorem 3.9.

Proof. Let then $\varphi\colon G \longrightarrow U_n(\mathbb{C})$, $\rho\colon G \longrightarrow U_m(\mathbb{C})$ be inequivalent irreducible unitary representations of G.

For *(i)*, let $A = E_{ki} \in M_{mn}(\mathbb{C})$. Then, as φ and ρ are inequivalent, Proposition 3.11 implies that $\overline{A} = 0$. Hence $\langle \varphi_{ij}, \rho_{kl} \rangle = (\overline{A})_{lj} = 0$, as claimed.

For *(ii)*, suppose first that $k = i$ and $l = j$. Let $A = E_{ii}$. Then

$$\langle \varphi_{ij}, \varphi_{ij} \rangle = (\overline{A})_{jj} = \left(\frac{\mathrm{Tr}(A)}{\deg(\varphi)}\mathrm{Id}_{\mathbb{C}^n}\right)_{jj} = \left(\frac{1}{n}\mathrm{Id}_{\mathbb{C}^n}\right)_{jj} = \frac{1}{n}$$

using first Proposition 3.13 and then Proposition 3.11*(ii)*. If now $k \neq i$ and $l = j$ for instance, let $A = E_{ki}$ and observe that

$$\langle \varphi_{ij}, \varphi_{kl} \rangle = (\overline{A})_{jj} = \left(\frac{\mathrm{Tr}(A)}{\deg(\varphi)}\mathrm{Id}_{\mathbb{C}^n}\right)_{jj} = 0$$

since $\mathrm{Tr}(A) = 0$. The remaining cases are handled similarly, and the theorem is proved. □

Renormalizing, we deduce also the next corollary.

Corollary 3.14. *Let $\varphi\colon G \longrightarrow U_n(\mathbb{C})$ be an irreducible unitary representation of degree n. Then the n^2 functions*

$$\left\{\sqrt{n}\varphi_{ij} : 1 \leq i, j \leq n\right\}$$

form an orthonormal set in $L(G)$.

An important consequence of Theorem 3.9 is that there are only finitely many equivalence classes of irreducible representations. Indeed, first recall that each equivalence class contains a unitary representation. As the entries of inequivalent unitary representations of G form an orthogonal set in $L(G)$, they form in fact a linearly independent set of vectors in $L(G)$. Thus there are at most $\dim(L(G)) = |G|$ equivalence classes of irreducible representations.

Additionally, if $\varphi^{(1)}, \dots, \varphi^{(s)}$ form a complete set of representatives of the equivalence classes of irreducible representations of G and $d_i = \deg(\varphi^{(i)})$, then the functions

$$\left\{\sqrt{d_k}\varphi^{(k)}_{ij} : 1 \leq k \leq s, 1 \leq i, j \leq d_k\right\}$$

form an orthonormal set in $L(G)$, and thus

$$s \leq d_1^2 + d_2^2 + \cdots + d_s^2 \leq |G|.$$

We shall see later on that the second inequality is in fact an equality, whereas the first inequality is an equality if and only if G is abelian.

4. Character theory and central functions

Definition 4.1. Let G be a finite group and $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ be a representation of G. Its *character*, denoted χ_φ, is the function $\chi_\varphi\colon G \longrightarrow \mathbb{C}$ defined as

$$\chi_\varphi(g) := \mathrm{Tr}(\varphi_g),\ g \in G.$$

Moreover, if φ is irreducible then χ_φ is called an *irreducible character*.

Note that if φ is one-dimensional, then $\chi_\varphi = \varphi$. We will therefore not make the distinction between one dimensional representations and their characters.

The character of a representation encodes many informations about the representation, the first one being its degree.

Proposition 4.2. *Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ be a representation of G. Then $\chi_\varphi(e) = deg(\varphi)$.*

Proof. $\chi_\varphi(e) = \mathrm{Tr}(\varphi(e)) = \mathrm{Tr}(\mathrm{Id}_V) = \dim(V) = \deg(\varphi)$. □

Also, equivalent representations have same characters.

Proposition 4.3. *Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$, $\rho\colon G \longrightarrow \mathrm{GL}(W)$ be representations G. If $\varphi \sim \rho$, then $\chi_\varphi = \chi_\rho$.*

Proof. From the assumption, there is an isomorphism $T\colon V \longrightarrow W$ so that $T\varphi_g = \rho_g T$ for any $g \in G$. It follows that

$$\chi_\varphi(g) = \mathrm{Tr}(\varphi_g) = \mathrm{Tr}(T^{-1}\rho_g T) = \mathrm{Tr}(T^{-1}T\rho_g) = \mathrm{Tr}(\rho_g) = \chi_\rho(g)$$

for any $g \in G$. □

The same proof shows that characters are constant on conjugacy classes.

Proposition 4.4. *Let $\varphi\colon G \longrightarrow \mathrm{GL}(V)$ be a representation of G. Then $\chi_\varphi(hgh^{-1}) = \chi_\varphi(g)$ for all $g, h \in G$.*

Proof. Let $g, h \in G$, and compute that

$$\chi_\varphi(hgh^{-1}) = \mathrm{Tr}(\varphi_{hgh^{-1}}) = \mathrm{Tr}(\varphi_h\varphi_g\varphi_{h^{-1}}) = \mathrm{Tr}(\varphi_h\varphi_{h^{-1}}\varphi_g) = \mathrm{Tr}(\varphi_g) = \chi_\varphi(g)$$

using that, inside Tr, matrices commute. □

Functions on the group that are constant on conjugacy classes play a key role in representation theory. They deserve their own name.

Definition 4.5. A function $f\colon G \longrightarrow \mathbb{C}$ is called *central* if $f(hgh^{-1}) = f(g)$ for any $g, h \in G$. The space of central functions is denoted $Z(L(G))$.

Equivalently, $f\colon G \longrightarrow \mathbb{C}$ is central if $f(gh) = f(hg)$ for any $g, h \in G$.

Proposition 4.6. *$Z(L(G))$ is a subspace of $L(G)$.*

Proof. Let $f_1, f_2 \in Z(L(G))$ and $c \in \mathbb{C}$. We have

$$(f_1 + cf_2)(hgh^{-1}) = f_1(hgh^{-1}) + cf_2(hgh^{-1}) = f_1(g) + cf_2(g) = (f_1 + cf_2)(g)$$

for any $g, h \in G$, whence $f_1 + cf_2 \in Z(L(G))$. □

As $Z(L(G))$ is a subspace, we can try to find its dimension. Let $\mathrm{CL}(G)$ be the set of conjugacy classes of G. For $C \in \mathrm{CL}(G)$, let

$$\begin{aligned} \delta_C\colon G &\longrightarrow \mathbb{C} \\ g &\longmapsto \begin{cases} 1 & \text{if } g \in C \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Lemma 4.7. *The set $B = \{\delta_C : C \in CL(G)\}$ is a basis of $Z(L(G))$. In particular, one has $dim(Z(L(G)) = |CL(G)|$.*

Proof. First of all, if $C \in \mathrm{CL}(G)$, then $\delta_C \in Z(L(G))$. If $f \in Z(L(G))$ and $f(C)$ denotes the value of f on the conjugacy class C, then

$$f = \sum_{C \in \mathrm{CL}(G)} f(C)\delta_C$$

whence B generates $Z(L(G))$.

Now, if $C \neq C' \in \mathrm{CL}(G)$, then

$$\langle \delta_C, \delta_{C'} \rangle = \frac{1}{|G|} \sum_{g \in G} \delta_C(g)\overline{\delta_{C'}(g)} = 0$$

since there is no $g \in G$ lying into two distinct conjugacy classes. Thus B is an orthogonal set of vectors in $Z(L(G))$, and in particular is a set of linearly independent vectors. □

Thanks to Schur's orthogonality relations (Theorem 3.9), we can deduce a similar statement on characters.

Theorem 4.8. *Let $\varphi\colon G \longrightarrow \mathrm{GL}_n(\mathbb{C})$, $\rho\colon G \longrightarrow \mathrm{GL}_m(\mathbb{C})$ be irreducible representations of G. Then $\langle \chi_\varphi, \chi_\rho \rangle = \begin{cases} 1 & \text{if } \varphi \sim \rho \\ 0 & \text{if } \varphi \nsim \rho \end{cases}$.*

Proof. Since any representation is equivalent to a unitary one (Proposition 2.9) and since equivalent representations have equal characters (Proposition 4.3), we may assume that φ and ρ are unitary. Thus it follows that

$$\langle \chi_\varphi, \chi_\rho \rangle = \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(\varphi_g)\overline{\mathrm{Tr}(\rho_g)} = \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{n} \varphi_{ii}(g) \sum_{j=1}^{m} \overline{\rho_{jj}(g)} = \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \varphi_{ii}, \rho_{jj} \rangle.$$

If $\varphi \nsim \rho$, then $\langle \varphi_{ii}, \rho_{jj} \rangle = 0$ by Theorem 3.9, whence $\langle \chi_\varphi, \chi_\rho \rangle = 0$.

If $\varphi \sim \rho$, then

$$\langle \chi_\varphi, \chi_\rho \rangle = \langle \chi_\varphi, \chi_\varphi \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \varphi_{ii}, \varphi_{jj} \rangle.$$

From Theorem 3.9, $\langle \varphi_{ii}, \varphi_{jj} \rangle \neq 0$ if and only if $j = i$, in which case we get

$$\langle \chi_\varphi, \chi_\rho \rangle = \sum_{i=1}^{n} \frac{1}{n} = 1$$

as claimed. This terminates our proof. □

Corollary 4.9. *There are at most $|CL(G)|$ equivalence classes of irreducible representations of G.*

Proof. From Theorem 4.8, inequivalent representations have distinct characters. As distinct characters of irreducible representations form an orthonormal set of $Z(L(G))$, they are linearly independent, and thus their number cannot exceed $\dim(Z(L(G)) = |\mathrm{GL}(G)|$. □

If V is a vector space, φ a representation of G on V and $m \geq 1$, we write mV for the direct sum of m copies of V, as well as $m\varphi$ for the direct sum of m copies of φ.

Let $\varphi^{(1)}, \ldots, \varphi^{(s)}$ be a complete set of representatives of irreducible representations of G. Denote $d_i := \deg(\varphi^{(i)})$.

Definition 4.10. Let φ be a representation of G. If $\varphi \sim m_1\varphi^{(1)} \oplus \cdots \oplus m_s\varphi^{(s)}$, then we call m_i the *multiplicity* of $\varphi^{(i)}$ in φ. If $m_i \geq 1$, we say $\varphi^{(i)}$ is an *irreducible constituent* of φ.

Note that if $\varphi \sim m_1\varphi^{(1)} \oplus \cdots \oplus m_s\varphi^{(s)}$, then $\deg(\varphi) = m_1d_1 + \cdots + m_sd_s$.

At this point, it is not clear that the multiplicity is well-defined, since we did not prove that the decomposition into irreducible ones is unique. We tackle the question by showing directly that multiplicities can be computed from characters. Since characters depend only on the equivalence class of the representation (cf. Proposition 4.3), the multiplicity of $\varphi^{(i)}$ will be the same no matter how we decompose φ.

We first record the following property of characters.

Lemma 4.11. *Let φ, ρ be representations of G. Then $\chi_{\varphi\oplus\rho} = \chi_\varphi + \chi_\rho$.*

Proof. For any $g \in G$, $(\varphi \oplus \rho)_g$ is the diagonal block matrix

$$\begin{pmatrix} \varphi_g & 0 \\ 0 & \rho_g \end{pmatrix}$$

and since the trace is the sum of diagonal elements, the claim follows. □

We can thus use orthogonality of irreducible characters to extract the coefficients in the decomposition of an arbitrary representation.

Theorem 4.12. *Let $\varphi^{(1)}, \ldots, \varphi^{(s)}$ be a complete set of representatives of the equivalence classes of irreducible representations of G. Let*

$$\varphi \sim m_1\varphi^{(1)} \oplus \cdots \oplus m_s\varphi^{(s)}.$$

Then $m_i = \langle \chi_\varphi, \chi_{\varphi^{(i)}} \rangle$. In particular, the decomposition of φ into irreducible constituents is unique and φ is determined up to equivalence by its character.

Proof. From Lemma 4.11 we have

$$\chi_\varphi = m_1\chi_{\varphi^{(1)}} + \cdots + m_s\chi_{\varphi^{(s)}}$$

whence

$$\langle \chi_\varphi, \chi_{\varphi^{(i)}} \rangle = \langle m_1\chi_{\varphi^{(1)}} + \cdots + m_s\chi_{\varphi^{(s)}}, \chi_{\varphi^{(i)}} \rangle = m_i$$

using Theorem 4.8. The second and third statements are consequences of Proposition 4.3. □

From this computation, we deduce a practical condition for checking a given representation is irreducible.

Corollary 4.13. *A representation φ of a group G is irreducible if and only if $\langle \chi_\varphi, \chi_\varphi \rangle = 1$.*

Proof. If $\varphi \sim m_1\varphi^{(1)} \oplus \cdots \oplus m_s\varphi^{(s)}$, then

$$\langle \chi_\varphi, \chi_\varphi \rangle = \sum_{i=1}^{s} m_i \langle \chi_\varphi, \chi_{\varphi^{(i)}} \rangle = \sum_{i=1}^{s} m_i^2$$

Since m_i is a natural integer for all $1 \leq i \leq s$, $\langle \chi_\varphi, \chi_\varphi \rangle = 1$ if and only if there is an index i so that $m_i = 1$, and $m_j = 0$ for $j \neq i$. This amounts to saying that $\varphi \sim \varphi^{(i)}$, which is equivalent to φ being irreducible by Proposition 2.4. □

Example 4.14. Consider the representation of $G = S_3$ given by

$$\varphi_{(12)} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \varphi_{(123)} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

The values of χ_φ are then $\chi_\varphi(\mathrm{Id}) = 2$, $\chi_\varphi((12)) = 0$, $\chi_\varphi((123)) = -1$, and Id, (12), (123) is a complete set of representatives of the conjugacy classes of G, whose cardinalities are respectively 1, 3 and 2. Thus

$$\langle \chi_\varphi, \chi_\varphi \rangle = \frac{1}{6}(1 \cdot 2^2 + 3 \cdot 0^2 + 2 \cdot (-1)^2) = 1$$

and φ is irreducible, as already showed in Example 1.16.

Example 4.15. Consider the standard representation φ of $G = S_4$ as in Example 1.5. G has five conjugacy classes, with representatives (1), (12)(34), (12), (1234), (123), of cardinalities 1, 3, 6, 6 and 8 respectively. Values of χ_φ are then $\chi_\varphi((1)) = 4$, $\chi_\varphi((12)(34)) = 0$, $\chi_\varphi((12)) = 2$, $\chi_\varphi((1234)) = 0$, $\chi_\varphi((123)) = 1$, whence

$$\langle \chi_\varphi, \chi_\varphi \rangle = \frac{1}{24}(1 \cdot 4^2 + 3 \cdot 0^2 + 6 \cdot 2^2 + 6 \cdot 0^2 + 8 \cdot 1^2) = 2$$

and φ is not irreducible. This agrees with the previous observation that $e_1 + e_2 + e_3 + e_4$ generates an invariant subspace of $\mathbb{C}^4$.

Example 4.16. We already know two irreducible representations of S_3: the trivial one and φ of Example 4.14. Observe that also the signature $\varepsilon \colon S_3 \longrightarrow \mathbb{C}^*$ is a group homomorphism, and therefore a one-dimensional representation of S_3. Since its character is itself, we directly compute

$$\langle \chi_\varepsilon, \chi_\varepsilon \rangle = \frac{1}{6}(1 \cdot 1^2 + 3 \cdot (-1)^2 + 2 \cdot 1^2) = 1$$

and thus ε is irreducible as well. As the values of its character differ from the values of the character of the trivial representation and the character of φ, ε is not equivalent to any of those representations. In fact, since S_3 has three conjugacy classes, Corollary 4.9 ensures it has at most three inequivalent irreducible representations, and since we found already three inequivalent irreducible representations, it has *exactly* three inequivalent irreducible representations.

Let us now decompose the standard representation ρ of S_3 (cf. Example 1.5) as a direct sum of $\mathbf{1}_{S_3}$, ε and φ. Indeed, we know that

$$\rho \sim m_1 \mathbf{1}_{S_3} \oplus m_2 \varepsilon \oplus m_3 \varphi.$$

Moreover, the character of ρ is given by $\chi_\rho((1)) = 3$, $\chi_\rho((12)) = 1$, $\chi_\rho((123)) = 0$. Using Theorem 4.12, we compute

$$m_1 = \langle \chi_\rho, \chi_{\mathbf{1}_{S_3}} \rangle = \frac{1}{6}(1 \cdot 3 \cdot 1 + 3 \cdot 1 \cdot 1 + 2 \cdot 0 \cdot 1) = 1$$
$$m_2 = \langle \chi_\rho, \chi_\varepsilon \rangle = \frac{1}{6}(1 \cdot 3 \cdot 1 + 3 \cdot 1 \cdot (-1) + 2 \cdot 0 \cdot 1) = 0$$
$$m_3 = \langle \chi_\rho, \chi_\varphi \rangle = \frac{1}{6}(1 \cdot 3 \cdot 2 + 3 \cdot 1 \cdot 0 + 2 \cdot 0 \cdot (-1)) = 1$$

and we conclude that $\rho \sim \mathbf{1}_{S_3} \oplus \varphi$.

Let $\mathbb{C}G$ denote the vector space of all formal linear combinations of elements of G, that is

$$\mathbb{C}G := \left\{ \sum_{g \in G} a_g g : a_g \in \mathbb{C} \right\}.$$

Addition and multiplication are defined using addition and multiplication in $\mathbb{C}$:

$$\left(\sum_{g \in G} a_g g \right) + \left(\sum_{g \in G} b_g g \right) := \sum_{g \in G} (a_g + b_g) g, \ \lambda \left(\sum_{g \in G} a_g g \right) := \sum_{g \in G} (\lambda a_g) g.$$

This vector space can be endowed with an inner product, given by

$$\left\langle \sum_{g \in G} a_g g, \sum_{g \in G} b_g g \right\rangle := \sum_{g \in G} a_g \overline{b_g}.$$

Definition 4.17. Let G be a finite group. Its *regular representation* is the group homomorphism $L \colon G \longrightarrow \mathrm{GL}(\mathbb{C}G)$ defined as

$$L_g \left(\sum_{h \in G} c_h h \right) := \sum_{h \in G} c_h (gh) = \sum_{x \in G} c_{g^{-1}x} x.$$

We will see below that the left regular representation is *never* irreducible, but has the feature of containing all irreducible representations of the group as subrepresentations.

To begin, we shall indeed prove it is a representation.

Lemma 4.18. $L \colon G \longrightarrow \mathrm{GL}(\mathbb{C}G)$ *is a unitary representation of* G.

Proof. Linearity of L_g for any $g \in G$ is straightforward to establish. Also if $g, h \in G$ then

$$L_g \left(L_h \left(\sum_{x \in G} a_x x \right) \right) = L_g \left(\sum_{x \in G} a_x (hx) \right) = \sum_{x \in G} a_x (ghx) = L_{gh} \left(\sum_{x \in G} a_x x \right)$$

so $L \colon G \longrightarrow \mathrm{GL}(\mathbb{C}G)$ is a group homomorphism. To prove L_g is unitary for any $g \in G$, it is enough to prove it preserves the inner product on $\mathbb{C}G$. We compute then

$$\begin{aligned}
\left\langle L_g \left(\sum_{h \in G} a_h h \right), L_g \left(\sum_{h \in G} b_h h \right) \right\rangle &= \left\langle \sum_{h \in G} a_h (gh), \sum_{h \in G} b_h (gh) \right\rangle \\
&= \left\langle \sum_{h \in G} a_{g^{-1}h} h, \sum_{h \in G} b_{g^{-1}h} h \right\rangle \\
&= \sum_{h \in G} a_{g^{-1}h} \overline{b_{g^{-1}h}} \\
&= \sum_{t \in G} a_t \overline{b_t} \\
&= \left\langle \sum_{h \in G} a_h h, \sum_{h \in G} b_h h \right\rangle
\end{aligned}$$

and thus L_g is unitary for all $g \in G$. This concludes the proof. □

The character of L takes a very simple form.

Proposition 4.19. *We have $\chi_L(g) = |G|\delta_e(g)$.*

Proof. Let $G = \{g_1, \ldots, g_n\}$ where $n = |G|$. Let $g \in G$. As G is a basis of $\mathbb{C}G$, we can compute the matrix $[L_g]$ of $L_g \colon \mathbb{C}G \longrightarrow \mathbb{C}G$ with respect to that basis. Since $L_g(g_i) = gg_i$, we have

$$([L_g])_{ij} = \begin{cases} 1 & \text{if } g_i = gg_j \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } g = g_i g_j^{-1} \\ 0 & \text{otherwise} \end{cases}.$$

In particular $([L_g])_{ii} = 1$ if $g = e$ and 0 otherwise, and it follows that

$$\chi_L(g) = \mathrm{Tr}([L_g]) = \sum_{i=1}^{n}([L_g])_{ii} = |G|\delta_e(g)$$

as claimed. □

We can now establish the decomposition of L into irreducible constituents.

Proposition 4.20. *Let $\varphi^{(1)}, \ldots, \varphi^{(s)}$ be a complete set of inequivalent irreducible unitary representations of G. For $1 \leq i \leq s$, denote $d_i = deg(\varphi^{(i)})$. Then one has*

$$L \sim d_1\varphi^{(1)} \oplus \cdots \oplus d_s\varphi^{(s)}.$$

Proof. Invoking Theorem 4.12, it suffices to compute $\langle \chi_L, \chi_{\varphi^{(i)}} \rangle$ to deduce the decomposition of L. But, thanks to Proposition 4.2 and Proposition 4.19, we have

$$\langle \chi_L, \chi_{\varphi^{(i)}} \rangle = \frac{1}{|G|}\sum_{g\in G}\chi_L(g)\overline{\chi_{\varphi^{(i)}}(g)} = \frac{1}{|G|}|G|\mathrm{deg}(\varphi^{(i)}) = d_i$$

for any $1 \leq i \leq s$, and the conclusion follows. □

We can then complete our previous results.

Proposition 4.21. *With the above notation, $|G| = d_1^2 + \cdots + d_s^2$.*

Proof. As $L \sim d_1\varphi^{(1)} \oplus \cdots \oplus d_s\varphi^{(s)}$, it follows from Proposition 4.3 and Lemma 4.11 that

$$\chi_L = d_1\chi_{\varphi^{(1)}} + \cdots + d_s\chi_{\varphi^{(s)}}.$$

Evaluating this equality at $g = e$ and using that $\chi_{\varphi^{(i)}}(e) = d_i$, $1 \leq i \leq s$, we get

$$|G| = \chi_L(e) = d_1\chi_{\varphi^{(1)}}(e) + \cdots + d_s\chi_{\varphi^{(s)}}(e) = d_1^2 + \cdots + d_s^2$$

as was to be shown. □

Corollary 4.22. *With the above notations, the set*

$$B = \left\{\sqrt{d_k}\varphi_{ij}^{(k)} : 1 \leq k \leq s, 1 \leq i, j \leq d_k\right\}$$

is an orthonormal basis of $L(G)$.

Proof. We already know that B form an orthonormal set of $L(G)$ (by the discussion following Corollary 3.14), and as

$$|B| = d_1^2 + \cdots + d_s^2 = |G| = \dim(L(G))$$

we deduce that B is a basis of $L(G)$. □

Here is another basis for the space $Z(L(G))$ of central functions on the group G.

Theorem 4.23. *Irreducible characters $\chi_1, \ldots, \chi_s$ form an orthonormal basis of $Z(L(G))$.*

Proof. From Theorem 4.8, we already know irreducible characters are pairwise orthogonal, and in particular linearly independent. It remains to show they generate $Z(L(G))$.

Let then $f \in Z(L(G))$. By Corollary 4.22, there exist $c_{ij}^{(k)}$ so that

$$f = \sum_{i,j,k} c_{ij}^{(k)} \varphi_{ij}^{(k)}.$$

Let $x \in G$. As f is central, we then have

$$\begin{aligned}
f(x) &= \frac{1}{|G|} \sum_{g \in G} f(gxg^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{i,j,k} c_{ij}^{(k)} \varphi_{ij}^{(k)}(gxg^{-1}) \\
&= \sum_{i,j,k} c_{ij}^{(k)} \frac{1}{|G|} \sum_{g \in G} \varphi_{ij}^{(k)}(gxg^{-1}) \\
&= \sum_{i,j,k} c_{ij}^{(k)} \left[\frac{1}{|G|} \sum_{g \in G} \varphi_g^{(k)} \varphi_x^{(k)} \varphi_{g^{-1}}^{(k)} \right]_{ij} \\
&= \sum_{i,j,k} c_{ij}^{(k)} \left[\overline{\varphi_x^{(k)}} \right]_{ij} \\
&= \sum_{i,j,k} c_{ij}^{(k)} \left[\frac{\mathrm{Tr}(\varphi_x^{(k)})}{d_k} \mathrm{Id} \right]_{ij} \\
&= \sum_{i,k} c_{ii}^{(k)} \frac{1}{d_k} \chi_k(x)
\end{aligned}$$

using Proposition 3.11 for the sixth equality. Hence $f = \sum_{i,k} c_{ii}^{(k)} \chi_k$, which shows that irreducible characters span $Z(L(G))$. The proof is complete. □

Since indicator functions of conjugacy classes also form a basis of $Z(L(G))$, we deduce the following.

Corollary 4.24. *The number of equivalence classes of irreducible representations of G equals the number of conjugacy classes of G.*

As a particular case of this corollary, we deduce that a group G is abelian if and only if it has $|G|$ irreducible representations, up to equivalence.

Example 4.25. Let $G = \mathbb{Z}/n\mathbb{Z}$, and $\omega = e^{\frac{2\pi i}{n}}$. Let $\chi_k\colon G \longrightarrow \mathbb{C}^*$, $[m] \longmapsto \omega^{km}$. Then $\chi_0, \ldots, \chi_{n-1}$ are the irreducible representations of G. Their values on conjugacy classes are usually gathered in a table, called the *character table* of G. For instance, the character table of $\mathbb{Z}/4\mathbb{Z}$ looks like

	[0]	[1]	[2]	[3]
χ_0	1	1	1	1
χ_1	1	-1	1	-1
χ_2	1	i	-1	$-i$
χ_3	1	$-i$	-1	i

Example 4.26. From Example 4.16, we see that the character table of $G = S_3$ is

	Id	(12)	(123)
χ_0	1	1	1
χ_ε	1	-1	1
χ_φ	2	0	-1

with $\varepsilon\colon S_3 \longrightarrow \mathbb{C}^*$ the signature and φ the irreducible 2−dimensional representation from Example 4.14.

In these two tables, we note that the inner product of two distinct columns is always 0. This is in fact always the case.

Theorem 4.27. *Let C, C' be conjugacy classes of G and $g \in C$, $h \in C'$. Then*

$$\sum_{i=1}^{s} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} \frac{|G|}{|C|} & \text{if } C = C' \\ 0 & \text{otherwise} \end{cases}.$$

In particular, columns of the character table are pairwise orthogonal.

Proof. Writing $\delta_{C'} = \sum_{i=1}^{s} \langle \delta_{C'}, \chi_i \rangle \chi_i$, we compute

$$\begin{aligned} \delta_{C'}(g) &= \sum_{i=1}^{s} \langle \delta_{C'}, \chi_i \rangle \chi_i(g) \\ &= \sum_{i=1}^{s} \frac{1}{|G|}\left(\sum_{x \in G} \delta_{C'}(x)\overline{\chi_i(x)}\right)\chi_i(g) \\ &= \sum_{i=1}^{s} \frac{1}{|G|}\left(\sum_{x \in C'} \overline{\chi_i(x)}\right)\chi_i(g) \\ &= \frac{|C'|}{|G|} \sum_{i=1}^{s} \overline{\chi_i(h)}\chi_i(g) \end{aligned}$$

and the claim follows. □

5. Representations of finite abelian groups

As shown in Example 4.25, we know all irreducible representations of $\mathbb{Z}/n\mathbb{Z}$. As any finite abelian group is a direct product of these, we only need to describe how to get irreducible representations of direct products of groups from irreducible representations of each factor. This is done by the next result.

Proposition 5.1. *Let G, H be abelian groups, and let $\chi_1, \ldots, \chi_n, \varphi_1, \ldots, \varphi_m$ be their irrducible representations (in particular $n = |G|$ and $m = |H|$). For $1 \le i \le n$, $1 \le j \le m$, define $\alpha_{ij}\colon G \times H \longrightarrow \mathbb{C}^*$ by*

$$\alpha_{ij}(g,h) = \chi_i(g)\varphi_j(h), \ (g,h) \in G \times H.$$

Then $\{\alpha_{ij} : 1 \le i \le n, 1 \le j \le m\}$ form a complete set of irreducible representations of $G \times H$.

Proof. Let $1 \le i \le n$, $1 \le j \le m$. For all $(g,h), (g',h') \in G \times H$, we have

$$\begin{aligned} \alpha_{ij}((g,h)(g',h')) &= \alpha_{ij}(gg', hh') \\ &= \chi_i(gg')\varphi_i(hh') \\ &= \chi_i(g)\varphi_i(h)\chi_i(g')\varphi_{i'}(h') \\ &= \alpha_{ij}(g,h)\alpha_{ij}(g',h') \end{aligned}$$

so $\alpha_{ij}\colon G \times H \longrightarrow \mathbb{C}^*$ is a group homomorphism, namely a 1-dimensional representation of $G \times H$. It follows in particular that α_{ij} is irreducible. We now show that the set mentioned in the statement has no repetitions, *i.e.* that $\alpha_{ij} = \alpha_{kl}$ implies $i = j$ and $k = l$. If $\alpha_{ij} = \alpha(kl)$, then

$$\chi_i(g) = \chi_i(g)\varphi_j(e_H) = \alpha_{ij}(g, e_H) = \alpha_{kl}(g, e_H) = \chi_k(g)$$

for any $g \in G$, whence $i = k$. Likewise, $j = l$. Lastly, since

$$|G \times H| = nm = |\{\alpha_{ij} : 1 \le i \le n, 1 \le j \le m\}|$$

we conclude that any irreducible representation of $G \times H$ is of the form α_{ij} for some $1 \le i \le n$ and $1 \le j \le m$. □

Example 5.2. The character table of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is given by

	([0],[0])	([1],[0])	([0],[1])	([1],[1])
α_{11}	1	1	1	1
α_{12}	1	1	−1	−1
α_{21}	1	−1	1	−1
α_{22}	1	−1	−1	1